MODULI SPACE OF PLANAR POLYGONAL LINKAGE: A COMBINATORIAL DESCRIPTION

GAIANE PANINA

ABSTRACT. We explicitly describe a structure of a CW-complex on the moduli space of a planar polygonal linkage. The cells of the maximal dimension are labeled by elements of the symmetric group. The entire construction is very much related (but not equal) to the combinatorics of the permutahedron. In particular, if the moduli space is a sphere, the CW-complex is dual to the permutahedron.

The dual complex is patched of Cartesian products of permutohedra and carries a natural PL-structure. It can be explicitly realized as as a surgery on the permutohedron.

As a by-product we prove that the space of convex configurations of a polygonal linkage is ball homeomorphic.

1. PRELIMINARIES AND NOTATION

A *polygonal n-linkage* is a sequence of positive numbers $L = (l_1, \dots, l_n)$. It should be interpreted as a collection of rigid bars of lengths l_i joined consecutively by revolving joints in a chain.

Definition 1.1. *A configuration* of L in the Euclidean plane $\mathbb{R}^2$ is a sequence of points $R = (p_1, \dots, p_{n+1})$, $p_i \in \mathbb{R}^2$ with $l_i = |p_i, p_{i+1}|$ and $l_n = |p_n, p_1|$ modulo the action of orientation preserving isometries of the space $\mathbb{R}^2$. We also call P *a closed chain* or a *polygon*.

A configuration carries a natural orientation which we indicate in figures by an arrow. The set $M(L)$ of all configurations modulo orientation preserving isometries is *the moduli space, or the configuration space* of the polygonal linkage L.

Throughout the paper we assume that no configuration of L fits a straight line. This assumption implies that the moduli space $M(L)$ is a closed manifold (see [2]). Clearly, its dimension equals $n - 3$.

Explicit descriptions of $M(L)$ existed for $n = 4, 5$, and 6 (see [2, 6]).

The aim of the preprint is to give an explicit combinatorial description of $M(L)$ as of a CW-complex. The cells of the dimension $n - 3$ are labeled by elements of the symmetric group S_{n-1}, and the entire construction is very much related (but not equal) to the combinatorics of the permutahedron.

Key words and phrases. Polygonal linkage, configuration space, moduli space, permutahedron, cyclic polytope.

We start with necessary preliminaries:

Convex configurations.

Definition 1.2. The set $M_{conv}(L)$ is the set of all convex configurations such that the orientation induced by the numbering goes counterclockwise.

The set $\overline{M}_{conv}(L)$ is the closure of $M_{conv}(L)$ in $M(L)$.

Lemma 1.3. (1) *The set $M_{conv}(L)$ is an open contractible subset of $M(L)$.*
(2) *The closure $\overline{M}_{conv}(L)$ is the suspension over the boundary $\partial M_{conv}(L)$.*

Proof. This was proven in [1]. A similar proof is as follows: clearly, $M_{conv}(L)$ is an open subset of $M(L)$. The oriented area function A (see [5]) has a unique critical point (the convex cyclic polygon) in the set $M_{conv}(L)$, where A attains its maximum. A gradient line never leaves the set $M_{conv}(L)$. Therefore $M_{conv}(L)$ can be contracted along the gradient lines, and the gradient lines provide the structure of the suspension. □

Later (in Proposition 2.10) we shall show that $M(L)_{conv}$ is homeomorphic to an open ball, and its closure $\overline{M}_{conv}(L)$ is homeomorphic to a closed ball of dimension $n-3$. Before we prove this let us reserve for $M(L)_{conv}$ and $\overline{M}_{conv}(L)$ the terms "an open cell" and "a closed cell". Indicating the dimension k of a cell, we write "a k-cell".

Polytopes. We shall use the combinatorial structure of the following polytopes:

The *permutohedron* Π_n (see http://en.wikipedia.org/wiki/Permutohedron) is defined as the convex hull of all points in $\mathbb{R}^n$ that are obtained by permuting the coordinates of the point $(1, 2, ..., n)$. The k-faces of Π_n correspond to ordered partitions of the set $\{1, 2, ..., n\}$ into $(n-k)$ non-empty parts.

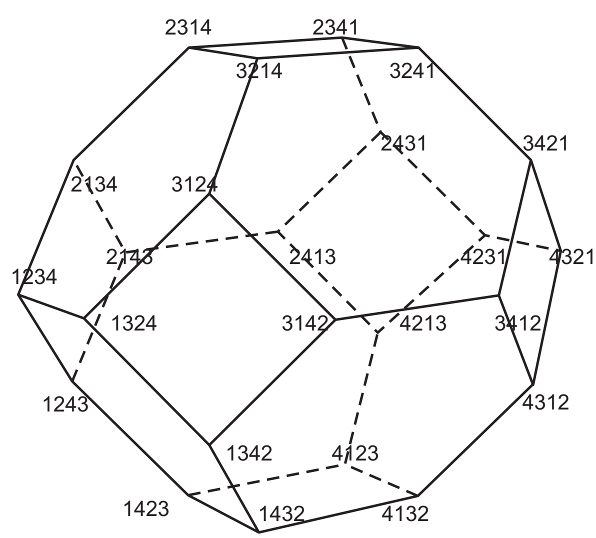

FIGURE 1. Permutahedron Π_4

The *cyclic polytope* $C(n,d)$ (see http://en.wikipedia.org/wiki/Cyclic_ polytope) is defined as the convex hull of n distinct points $p_1,...,p_n$ on the moment curve in $\mathbb{R}^d$. Its combinatorics is completely defined by the following property (Gale evenness condition): a d-subset $F \subset \{p_1,...,p_n\}$ forms a facet of $C(n,d)$ iff any two elements of $\{p_1,...,p_n\} \setminus F$ are separated by an even number of elements from F in the sequence $p_1,...,p_n$.

2. Structure of a CW-complex

Definition 2.1. A partition of $L=(l_1,\dots,l_n)$ is called *admissible* if the total length of any part does not exceed the total length of the rest.

Instead of partitions of $L=(l_1,\dots,l_n)$ we shall speak of partitions of the symbols $\{1,2,...,n\}$, keeping in mind the lengths l_i.

In the terminology of paper [3], all parts of an admissible partition are *short sequences.*

Definition 2.2. Two edges p_ip_{i+1} and p_jp_{j+1} of a configuration P are called *parallel* if the vectors $\overrightarrow{p_ip}_{i+1}$ and $\overrightarrow{p_jp}_{j+1}$ are parallel and codirected.

For instance, in Fig. 3 the blue and the red edges are parallel.

Given a configuration P of L **without parallel edges**, there exists a unique convex polygon $\overline{P}$ such that

(1) The edges of P are in one-to-one correspondence with the edges of $\overline{P}$. The bijection preserves the directions of the vectors.
(2) The induced orientations the edges of $\overline{P}$ give the counterclockwise orientation of $\overline{P}$.

The edges of the polygon $\overline{P}$ are the edges of P ordered by the slope (see Fig. 2). Obviously, $\overline{P} \in M_{conv}(\sigma L)$ for some permutation $\sigma \in S_n$. The permutation is defined up to some power of the cyclic permutation $(2,3,4,...,n,1)$. We identify it with an element of S_{n-1}.

Conversely, each convex polygon from $M_{conv}(\sigma L)$ is the image of some element of $M(L)$ under the above rearrangement mapping.

If P **has parallel edges**, a permutation which makes P convex is not unique one can choose any ordering on the set of parallel edges, see Fig. 3.

Thus each element of $M(L)$ is assigned a combinatorial object (we call it "*the labeling* "): a cyclically ordered partition of the set $\{1,2,3,4,...,n\}$. It is convenient to write such a partition as a (linearly ordered) string of sets where the set containing 1 is on the first place.

As usual, we assume that there is no ordering inside a set, that is, we identify two such objects whenever they differ on permutations of the elements inside the parts. For instance,

$$(\{1\}\{4625\}\{3\}) \neq (\{1\}\{3\}\{4625\}) = (\{1\}\{3\}\{2465\}) \neq (\{13\}\{2465\}).$$

Two points from $M(L)$ (that is, two configurations) are said to be *equivalent* if they have one and the same labeling. The moduli space $M(L)$ is thus the union of all equivalence classes. The equivalence classes are open cells.

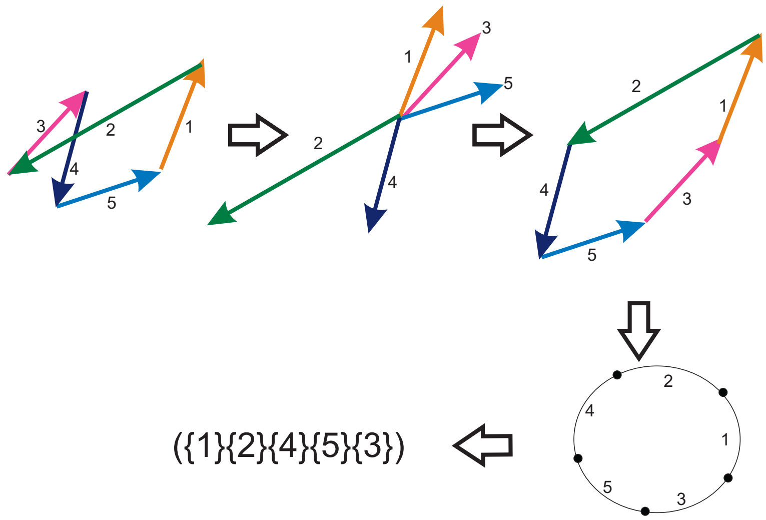

FIGURE 2. Rearrangment: no parallel edges

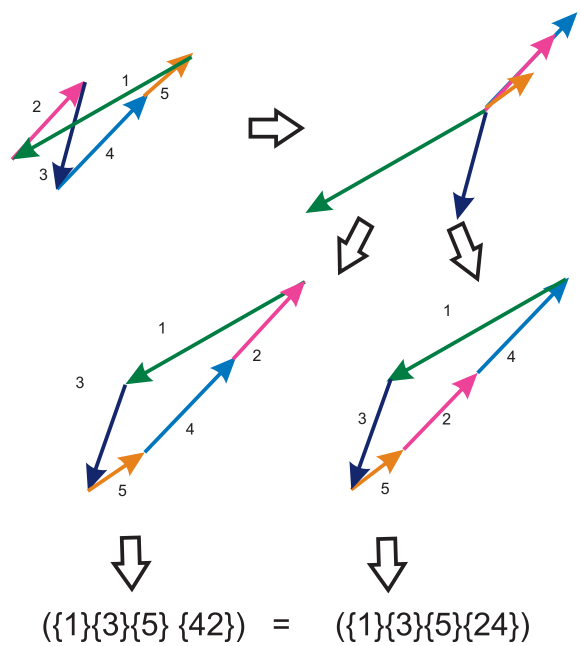

FIGURE 3. Rearrangment with parallel edges

We have proven the following theorem:

Theorem 2.3. *The collection of equivalence classes yields a structure of a CW-complex $CWM(L)$ on the moduli space $M(L)$.*

(1) *Open k-cells of the complex $CWM(L)$ are labeled by cyclically ordered admissible partitions of the set $\{1, 2, ..., n\}$ into $k+3$ non-empty parts.*
(2) *In particular, the facets of the complex (that is, cells of maximal dimension $n-3$) are labeled by cyclic orderings of the set $\{1, 2, ..., n\}$. We identify them with the elements of the symmetric group S_{n-1}.*
(3) *The complex is regular, that is, the closure of each open set is ball homeomorphic.*
(4) *A (closed) cell C belongs to the boundary of some other (closed) cell C' iff the labeling of C is finer than the labeling of C'.*
(5) *The intersection of two (closed) cells with labels λ and μ is labeled by the coarsest ordered admissible partition ν such that both partition λ and μ refine ν. If such a partition ν does not exist, the intersection is empty.*
(6) *The star of any k-dimensional face is combinatorially equivalent to the Cartesian product of $k+3$ permutohedra.*

The proof follows directly from the above construction. It remains to prove that the cells are homeomorphic to balls. This will be done in Proposition 2.10. □

Example 2.4. *Let $n = 4$; $\;l_1 = l_2 = l_3 = 1,\ l_4 = 1/2$. The moduli space $M(L)$ is a disjoint union of two circles, and the cell complex $CWM(L)$ is as depicted in Fig. 4.*

Example 2.5. *Assume that $l_2 = l_3 = ... = l_n$; $\;l_2 + l_3 + ... + l_n = l_1 + \varepsilon$ where ε is small. Then $M(L)$ is a $(n-3)$-sphere, and the complex $CWM(L)$ is dual to the boundary complex of the permutahedron Π_{n-1}.*

Proof. Indeed, each admissible partition is of the type

$$(\{1\},\ \text{any ordered partition of the rest}).$$

This means that the facets of $CWM(L)$ are in a natural bijection with the vertices of Π_{n-1}. It remains to observe that the patching rules for $CWM(L)$ are exactly dual to the combinatorics of the permutohedron. □

Example 2.6. *Let $n = 5$, $L = (1, 1, 1, 1, 1)$. Then $CWM(L)$ is a surface of genus 4 patched of 24 pentagons. The CW-complex is completely transitive. This means that combinatorial equivalence of any two pentagons extends to an automorphism of the entire CW-complex.*

Example 2.7. *Let $n = 2k+1$, $L = (1, 1, ..., 1)$. Then $CWM(L)$ is patched of $(2k)!$ copies of duals to the cyclic polytope $C(n, n-3)$. However, unlike the previous example, the CW-complex is not completely transitive, just facet-transitive: for every two facets there exists an automorphism of the complex mapping one face to the other.*

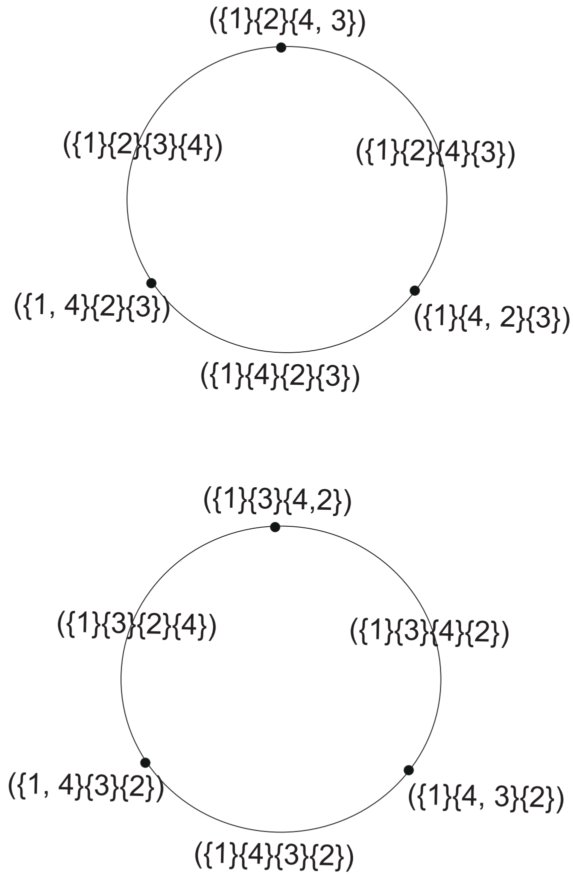

FIGURE 4. Moduli space of a 4-gonal linkage $(1, 1, 1, 1/2)$

Proof. Fix one facet of $CWM(L)$. Without loss of generity we may assume that it corresponds to the permutation $(1, 2, ..., n)$. The following "starlike" bijection φ which maps the vertices $x_1, ..., x_n$ of the cyclic polytope $C(n, n-3)$ to facets of the cell $(\{1\}\{2\}\{3\}\{4\}...\{n\})$ is a combinatorial duality:

$$\varphi(x_1) = (\{1, 2\}\{3\}\{4\}...\{n\}),$$
$$\varphi(x_2) = (\{1\}\{2\}...\{k+1, k+2\}...\{n\}),$$
$$\varphi(x_3) = (\{1, n\}\{2\}...\{n-1\}),$$
$$\varphi(x_4) = (\{1\}\{2\}...\{k, k+1\}...\{n\}),$$
$$\varphi(x_5) = (\{1\}\{2\}...\{n-1, n\}),$$
$$... \quad \square$$

Example 2.8. *Let $L' = (l_1, l_2, ..., l_n, \varepsilon)$, where ε is small. It is known that $M(L') = M(L) \times S^1$. On the one hand, this does not extend to CW-complex structure. On the other hand, there is a natural forgetting projection*

$$\pi : CWM(L') \to CWM(L)$$

which removes the number $n+1$ from the labeling. Fig. 5 depicts the local structure of π for the case $L = (1, 1, 1, 1/2)$.

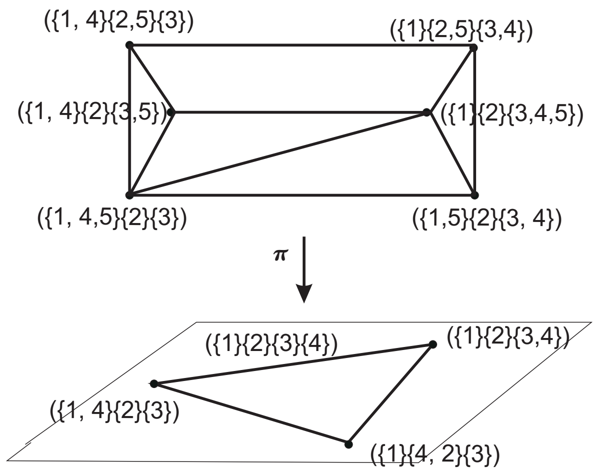

FIGURE 5. We depict the preimage of the edge $(\{1\}\{2\}\{3\}\{4\})$

Proposition 2.9. *The boundary of each closed cell of $CWM(L)$ is combinatorially equivalent to a face of the cyclic polytope $C(D+3, D)$ for some D.*

Proof. We may assume that all l_i are integers, and that their sum is odd. The space $M(L)$ embeds in a natural way in the space

$$M(\underbrace{1, 1, ..., 1}_{D}), \qquad \text{where} \qquad D = \sum_{i=1}^{n} l_i.$$

The embedding maps a polygon with edgelengths $l_1, ..., l_n$ to the equilateral polygon with first l_1 edges aligned, next l_2 edges aligned, etc. The embedding respects the structure of CW-complex. In other words, $CWM(L)$ is a subcomplex of the complex described in Example 2.7. □

Proposition 2.10. *Each closed k-cell is homeomorphic to a k-dimensional ball.*

Proof goes by induction. For small dimensions the statement is true. By Lemma 1.3, for dimension $k+1$, a closed cell C is a suspension of a manifold which is patched up of k-cells in a way which follows the combinatorics of a convex polytope (Proposition 2.9). By inductive assumption, all those k-cells are k-balls. Therefore the boundary ∂C is sphere homeomorphic. Since C is the suspension of ∂C, the closed cell C is a closed ball. □

Theorem 2.11. *The dual cell complex $CWM^*(L)$ carries a natural structure of a PL-manifold.*

Proof. Indeed, the dual complex $CWM^*(L)$ is patched of Cartesian products of permutohedra (see Theorem 2.3). For each facet of $CWM^*(L)$ we take the Cartesian product of three standard permutohedra. The faces that are identified via the patching rule are isometric. □

3. Generic construction: surgery on permutohedron

Fix an n-linkage L and consider the set of vertices of the permutohedron $\Pi_{n-1} \subset \mathbb{R}^{n-2}$.

We assume that the faces are labeled by ordered partitions of the set $(1, ..., n-1)$. We introduce the following mapping φ: each vertex of $CWM^*(L)$ (which is a cyclically ordered set $\{1, ..., n\}$) we map by φ to the vertex of $\Pi_{n-1} \subset \mathbb{R}^{n-2}$ by cutting the cyclic ordering at the position of "n" and omitting "n" from the label.

Lemma 3.1. *Whenever $v_1, ..., v_k$ are the vertices of a k-face of $CWM^*(L)$, their images $\varphi(v_1), ..., \varphi(v_k)$ belong to a k-plane in $\mathbb{R}^{n-2}$.* □

Due to the lemma we can patch an appropriate polytope for each face of $CWM^*(L)$.

Some of these polytopes are former faces of Π_{n-1}; some are new "diagonal" polytopes.

Example 3.2. *Let $n = 5$, $L = (2,9; 1; 1; 1; 0,001)$. The algorithm for the moduli space of 4-gonal linkage starts with the permutahedron Π_4 (see Fig. 6). The two shadowed faces are removed and replaced by six blue ones.*

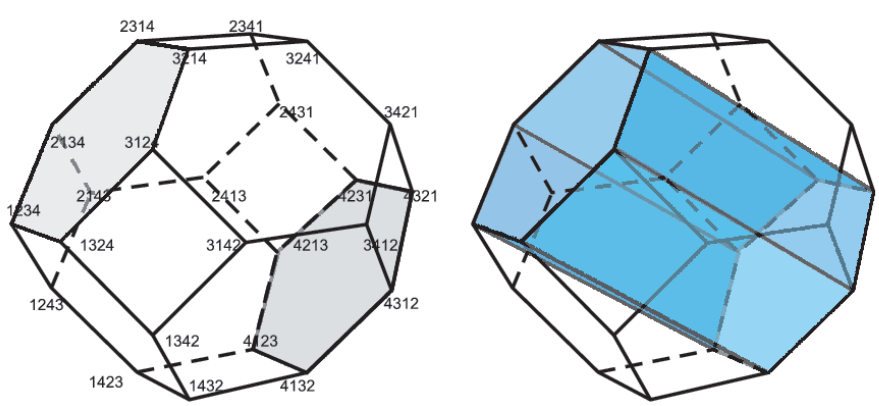

Figure 6. $CWM^*(L)$ for a 5-gonal linkage $L = (2,9; 1; 1; 1; 0,001)$: remove from the permutahedron the grey facets and patch in the blue cylinder.

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St. Petersburg Institute for Informatics and Automation RAS; St. Petersburg State University, e-mail:gaiane-panina@rambler.ru